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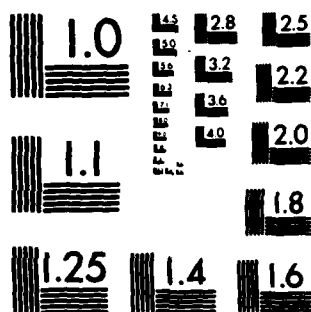
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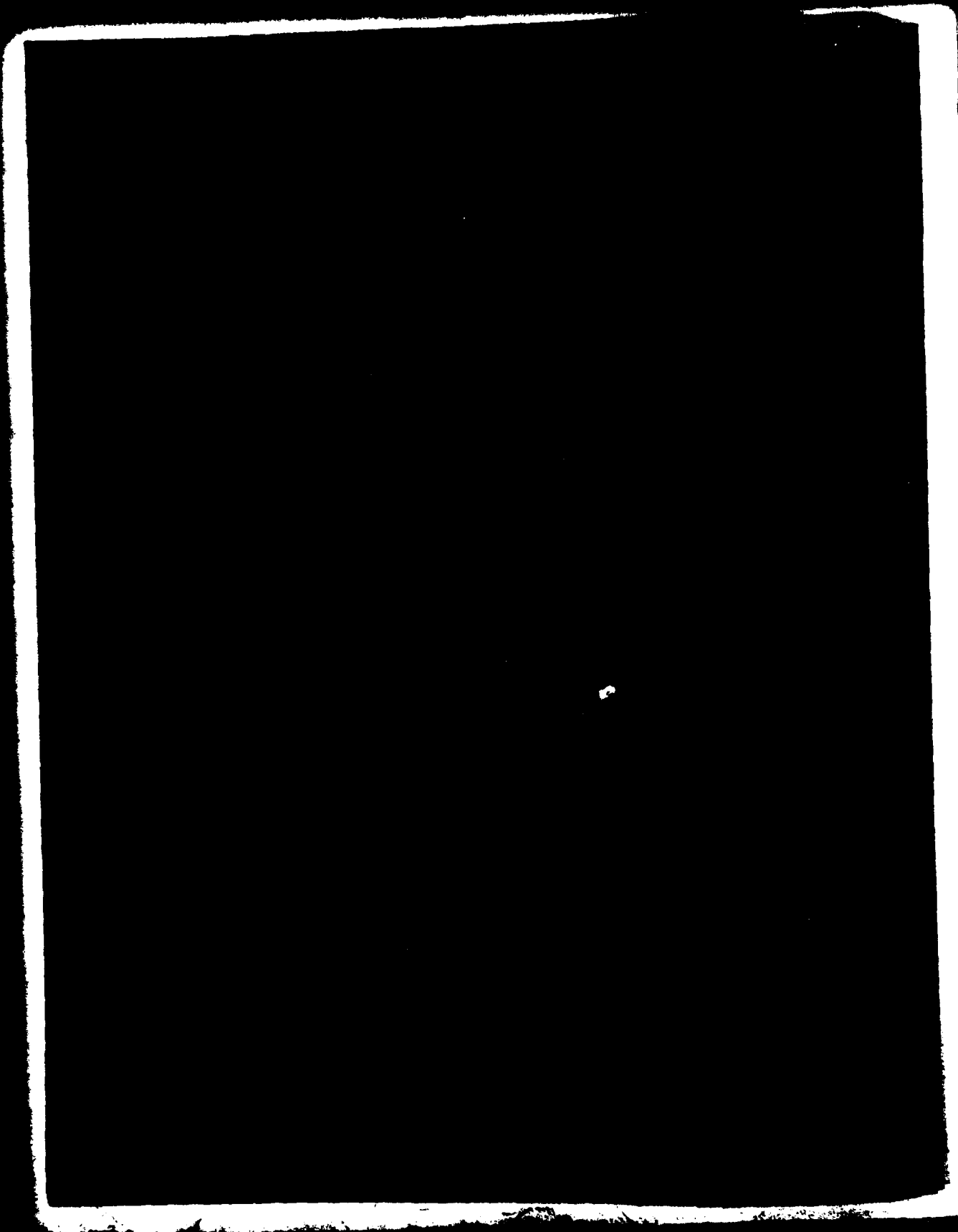
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A procedure for deriving the Lanczos Coordinates is explained and their use in structural dynamics analysis as an alternative to modal coordinates is discussed. The coordinates are obtained by an inverse iteration procedure in which orthogonality is imposed between the vectors resulting from successive iteration cycles. Using these Lanczos coordinates the equations of motion are transformed to tridiagonal form, which provides for very efficient time-stepping solution. The effectiveness of the method is demonstrated by a numerical example.			

# **DYNAMIC ANALYSIS OF STRUCTURES USING LANCZOS COORDINATES**

*by*

**Bahram Nour-Omid<sup>†</sup>**

**and**

**Ray W. Clough<sup>‡</sup>**

## **SUMMARY**

A procedure for deriving the Lanczos Coordinates is explained and their use in structural dynamics analysis as an alternative to modal coordinates is discussed. The coordinates are obtained by an inverse iteration procedure in which orthogonality is imposed between the vectors resulting from successive iteration cycles. Using these Lanczos coordinates the equations of motion are transformed to tridiagonal form, which provides for very efficient time-stepping solution. The effectiveness of the method is demonstrated by a numerical example.

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## Introduction

Analysis of the response of structures subjected to dynamic loads typically involves first establishing a discretized model of the structural system, using an appropriate discretization procedure such as finite elements or finite differences. Then the equations of motion are expressed as follows:

$$M\ddot{u} + C\dot{u} + Ku = f \quad (1)$$

in which  $M$ ,  $C$ ,  $K$ , are the  $n \times n$  mass, damping and stiffness matrices, respectively;  $f$  is the time dependent external load; and  $u$  is the displacement vector describing the response of the structure. The number of displacement coordinates,  $n$ , employed in the discretization depends on the complexity of the structure and on the amount of detail desired in the description of the dynamic stress response.

Because the discretized model of a complicated structural system may have many hundreds or even thousands of degrees of freedom, it is customary to reduce the equations of motion to a much smaller number before the dynamic response is calculated. In the analysis of linear structures, it has become essentially standard practice to express the response in terms of the undamped free vibration mode shapes, using only enough of the lower modes to express the behavior adequately. The main analytical problem then becomes the evaluation of the mode shapes, and the problem of reducing the number of degree of freedom is transferred to this phase of the analysis.

The Rayleigh-Ritz method [1] has been used widely to reduce the dimension of the equations of motion. This may reduce the dimensions of the equations of motion. This may be looked upon as nothing more than a coordinate transformation, the essential step being the choice of a set of vectors  $Y_n = [y_1, y_2, y_3, \dots, y_n]$  for describing the dynamic response. Thus an approximation to the dynamic response,  $u_n$ , is expressed as a linear combination of the chosen displacement vectors

$$u_n = Y_n x(t) \quad (2)$$

in which the amplitudes of these vectors,  $x(t)$ , are the generalized coordinates representing the approximate solution. Applying this coordinate transformation and its time derivatives to Eq. 1, leads to

$$\bar{M}_n \ddot{u} + \bar{C}_n \dot{u} + \bar{K}_n u = \bar{f}_n \quad (3)$$

in which the generalized coordinate mass, damping, and stiffness matrices and the external force vector are defined as follows:

$$\begin{aligned} \bar{M}_n &= Y_n^T M Y_n \\ \bar{C}_n &= Y_n^T C Y_n \\ \bar{K}_n &= Y_n^T K Y_n \\ \bar{f}_n &= Y_n^T f \end{aligned} \quad (4)$$

The basic problem in the use of the Rayleigh-Ritz method is that the quality of the approximate solution,  $u_n$ , depends entirely on the adequacy of the assumed set of displacement vectors,  $Y_n$ . One approach to dealing with this problem is to apply iterative "improvement" of the vectors. In this method, known as "subspace" or "simultaneous" iteration, the trial vectors are all subjected to inverse iteration combined with some technique (such as Gram-Schmidt orthogonalization) that forces convergence to independent shapes. Actually the convergence is to the lowest undamped vibration mode shapes,  $s_n$  and then the coordinate transformation

$$u_n = Z_n x(t) \quad (5)$$

leads to an uncoupled set of modal equations. That is, the equations equivalent to Eq. 3 are independent because the mode shape orthogonality properties applied to Eq. 4 leads to diagonal generalized mass, damping and stiffness matrices (assuming  $C$  is a proportional damping matrix).

It frequently is assumed that this modal coordinate transformation provides the most efficient method of dynamic response analysis because the independent modal equations can be solved separately and the total response obtained by superposition. However, there is no reason to believe that the modal coordinates will always give the best results with the fewest degrees of freedom. In fact, alternatives or supplements to mode superposition, such as the mode acceleration methods [2] and static corrections [3], have been developed to obtain satisfactory results with fewer coordinates. It is the purpose of this paper to describe how Lanczos coordinates can lead to



a still more efficient procedure for dynamic response analysis.

### Lanczos Algorithm

The Lanczos method [4], first introduced in 1950, is an efficient algorithm for extracting some frequencies and mode shapes of an eigensystem. It can be thought of as a means of constructing an orthogonal set of vectors, known as Lanczos vectors, for use in the Rayleigh-Ritz [1,7] procedure. Recently Wilson, et al [5] independently developed the idea of generating a set of orthogonal vectors for use in a Ritz type analysis as an alternative to the mode superposition method. Their method is identical to the Lanczos algorithm with full reorthogonalization [6,7]. The results they report show that these "Ritz vectors" give better results than the same number of modal coordinates.

It will be shown here that the Lanczos vectors can be used to formulate a very efficient means of dynamic response analysis without solving the vibration eigenproblem. These vectors do not have the full uncoupling property of the mode shapes, but they are much less expensive to generate. Moreover, they are derived from the applied load vector and include the static displaced shape as the first vector; thus no "static correction" is required no matter what the shape of the loading is nor how few vectors are employed in the analysis. Moreover, an indication of the number of vectors required to obtain the desired degree of accuracy may be determined during the derivation of the Lanczos vectors.

The Lanczos algorithm is closely related to the inverse iteration and power methods for calculating a single vibration mode shape and frequency of an eigen problem. Given a pair of matrices  $K$  and  $M$ , and a starting vector  $r$  these methods generates a sequence of vectors,  $\{r, K^{-1}Mr, (K^{-1}M)^2r, \dots, (K^{-1}M)^j r\}$ , during  $j$  iterations. These vectors are referred to as the Krylov sequence; the sequence converges to the eigenvector corresponding to the smallest eigenvalue, in magnitude, of the eigenproblem  $(K - \lambda M)s = 0$ .

The basic difference between the Lanczos method and the other two methods is that the information contained in each successive vector of the Krylov sequence is employed in the

dynamic response analysis, instead of using only the final converged vector. To be more specific, the Lanczos algorithm involves supplementing the Krylov sequence with a Gram-Schmidt orthogonalization process at each step; the result is a set of  $M$ -orthonormal vectors that is used to reduce the dimension of the dynamic equation set. In the form of the Lanczos algorithm described here, orthogonalization is applied only with respect to two preceding vectors, leading to a tridiagonal form of the dynamic equations that can be used to great advantage either directly in time step integration or in solution of the free vibration eigenproblem.

To derive the Lanczos algorithm it will be assumed for the moment that the first  $j$  Lanczos vectors,  $(q_1, q_2, \dots, q_j)$  have been found, and the analysis of the  $j + 1$  vector will be performed. The resulting vectors all satisfy the condition  $q_i^T M q_j = \delta_{ij}$ , where  $\delta_{ij}$  is the Kroneker delta; that is the vectors are orthonormal with respect to the mass matrix. To calculate  $q_{j+1}$ , a preliminary vector  $F_j$  is first calculated from the previous vector,  $q_j$ , as in the Krylov sequence:

$$F_j = K^{-1} M q_j \quad (6)$$

Now in general it may be assumed that this preliminary vector contains components from each of the preceding vectors, thus

$$F_j = r_j + \alpha_j q_j + \beta_j q_{j-1} + \gamma_j q_{j-2} + \dots \quad (7)$$

where  $r_j$  is the "pure" vector orthogonal to all previous Lanczos vectors, and  $\alpha_j, \beta_j, \gamma_j, \dots$  are the amplitudes of the previous vectors contained in  $F_j$ . These amplitude coefficients are evaluated from the orthonormality of the Lanczos vectors. Thus if both sides of Eq. 7 are multiplied by  $q_j^T M$ , the result is

$$q_j^T M F_j = q_j^T M r_j + \alpha_j q_j^T M q_j + \beta_j q_j^T M q_{j-1} + \gamma_j q_j^T M q_{j-2} + \dots \quad (8)$$

Here the first term on the right hand side vanishes by definition and all terms beyond the second vanish similarly due to orthogonality. The normalizing definition applied to the second term then reduces Eq. 8 to an expression for the amplitude of  $q_j$  along  $F_j$ :

$$\alpha_j = q_j^T M F_j \quad (9)$$

The amplitude of  $q_{j-1}$  contained in  $F_j$  may be found similarly by multiplying Eq. 7 by  $q_{j-1}^T M$ . In this case all terms except the third vanish by orthogonality and the coefficient of  $\beta_j$  is

unity, so

$$\beta_j = \mathbf{q}_{j-1}^T \mathbf{M} \mathbf{F}_j$$

But substituting Eq. 6 this gives  $\beta_j = \mathbf{q}_{j-1}^T \mathbf{M} \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j$ , and applying the transpose of Eq. 6 to the  $\mathbf{q}_{j-1}^T$  vector gives

$$\beta_j = \mathbf{F}_{j-1}^T \mathbf{M} \mathbf{q}_j \quad (10)$$

Finally, expanding  $\mathbf{F}_{j-1}$  in terms of its pure vector,  $\mathbf{r}_{j-1}$ , and the preceding Lanczos vectors, as in Eq. 7, the transpose of Eq. 10 becomes

$$\beta_j = \mathbf{q}_j^T \mathbf{M} \mathbf{r}_{j-1} + \alpha_{j-1} \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-1} + \beta_{j-1} \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-2} + \gamma_{j-1} \mathbf{q}_j^T \mathbf{M} \mathbf{q}_{j-3} + \dots \quad (11)$$

It is evident that all terms except the first vanish on the right hand side. Now  $\mathbf{q}_j$  is the vector obtained by normalizing  $\mathbf{r}_{j-1}$  with  $\beta_j$  as the normalizing factor, i.e.

$$\mathbf{q}_j = \frac{1}{\beta_j} \mathbf{r}_{j-1} \quad (12)$$

then the value of  $\beta_j$  is given by Eq. 11 :  $\beta_j = \frac{1}{\beta_j} \mathbf{r}_{j-1}^T \mathbf{M} \mathbf{r}_{j-1}$  or

$$\beta_j^2 = \mathbf{r}_{j-1}^T \mathbf{M} \mathbf{r}_{j-1} \quad (13)$$

Continuing in the same way, the amplitude of  $\mathbf{q}_{j-2}$  contained in  $\mathbf{F}_j$  is found to be

$$\gamma_j = \mathbf{q}_{j-2}^T \mathbf{M} \mathbf{F}_j \quad (14)$$

Following the procedure used to derive Eq. 11, this leads to

$$\gamma_j = \mathbf{q}_{j-2}^T \mathbf{M} \mathbf{r}_{j-2} + \alpha_{j-2} \mathbf{q}_{j-2}^T \mathbf{M} \mathbf{q}_{j-2} + \beta_{j-2} \mathbf{q}_{j-2}^T \mathbf{M} \mathbf{q}_{j-3} + \gamma_{j-2} \mathbf{q}_{j-2}^T \mathbf{M} \mathbf{q}_{j-4} + \dots \quad (15)$$

But using the normalizing relationship equivalent to Eq. 12,  $\mathbf{r}_{j-2} = \beta_{j-1} \mathbf{q}_{j-1}$ , hence when this is substituted into Eq. 15 it is clear that all terms on the right hand side vanish, with the result that  $\gamma_j = 0$ . A corresponding procedure could be used to demonstrate that all further terms in the expansion for  $\mathbf{F}_j$  (Eq. 7) vanish; in other words, the orthogonalization procedure used in generating each Lanczos vector need be applied only to the previous two vectors.

In summary, the Lanczos algorithm to derive the vector  $\mathbf{q}_{j+1}$ , may be expressed by the following sequence of equations:

$$\begin{aligned}
 \mathbf{F}_j &= \mathbf{K}^{-1} \mathbf{M} \mathbf{q}_j & (a) \\
 \mathbf{r}_j &= \mathbf{F}_j - \alpha_j \mathbf{q}_j - \beta_j \mathbf{q}_{j-1} & (b) \\
 \text{where : } \alpha_j &= \mathbf{q}_j^T \mathbf{M} \mathbf{F}_j & (c) \\
 \beta_j &= (\mathbf{r}_{j-1}^T \mathbf{M} \mathbf{r}_{j-1}) & (d) \\
 \mathbf{q}_{j+1} &= \frac{1}{\beta_{j+1}} \mathbf{r}_j & (e) \\
 \text{where : } \beta_j &= (\mathbf{r}_j^T \mathbf{M} \mathbf{r}_j) & (f)
 \end{aligned}
 \tag{16}$$

### Starting Vector

In general, the starting vector  $\mathbf{q}_1$  for the Lanczos algorithm may be chosen arbitrarily. Then setting  $\mathbf{q}_0 = 0$ , the procedure defined by Eqs. 16 will produce the second Lanczos vector  $\mathbf{q}_2$ , and the process may be continued to produce as many vectors as are desired. It will be noted that the cost of evaluating each successive vector is constant after  $\mathbf{q}_2$  has been obtained because it is necessary to impose orthogonality only with respect to the two preceding vectors.

The Lanczos algorithm is particularly advantageous in dynamic analysis when the applied load is of the form

$$\mathbf{f} = \mathbf{b} \epsilon(t) \tag{17}$$

that is, when the distribution of the load,  $\mathbf{b}$  remains constant and only its amplitude varies with time. The starting vector in this case is taken in the direction of the static displacement, thus

$$\mathbf{r}_0 = \beta_1 \mathbf{q}_1 = \mathbf{K}^{-1} \mathbf{b} \tag{18}$$

This choice gives the Lanczos vectors the important advantage noted before, that they automatically include the static displacement and avoid any possible need for a static correction. It is of interest to note that Wilson, et al. [5] adopted this starting vector for this reason. Further, it is evident from Eqs. 16 that if  $\mathbf{b}$  is orthogonal to any of the modes then all the Lanczos vectors will also be orthogonal to this mode.

An important type of dynamic loading of the form of Eq. 17 is earthquake excitation, where the effective loading results from accelerations applied at the supports. In this case,  $\mathbf{b} = \mathbf{M} \mathbf{s}$  where  $\mathbf{s}$  is the influence coefficient vector listing the displacement of the degrees of freedom due to

a unit support displacement, and  $\epsilon(t)$  is the acceleration history applied at the support.

### Dynamic Response Analysis Procedure

The orthogonality properties identified in the derivation of the Lanczos algorithm can be utilized effectively in transforming the equations of motion to a reduced form. Equating the two expression for  $r_j$ , given by Eqs. 16b and 16e, and substituting Eq. 16a leads to

$$r_j = \beta_{j+1} q_{j+1} = K^{-1} M q_j - q_j \alpha_j - q_{j-1} \beta_j \quad (19)$$

Transferring the right side to the left, the combined set of the Lanczos vectors obtained after  $m$  steps can be arranged in a single equation as follows:

$$\begin{bmatrix} K^{-1} M \end{bmatrix} \begin{bmatrix} Q_m \end{bmatrix} - \begin{bmatrix} Q_m \end{bmatrix} \begin{bmatrix} T_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} = r_m e_m^T \quad (20)$$

where  $Q_m$  is the  $n \times m$  matrix with columns  $q_1, q_2, \dots, q_m$ ,  $e_m$  is the last column of the  $I_m$ , identity matrix of dimension  $m$ , and  $T_m$  is a tridiagonal matrix made up of the coefficient  $\alpha_j, \beta_j$ :

$$T_m = \begin{bmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & \beta_3 & & \ddots & \\ & & & \ddots & \beta_m \\ & & & & \beta_m & \alpha_m \end{bmatrix} \quad (21)$$

Now multiplying Eq. 20 by  $Q_m^T M$  and applying the orthonormal properties of the Lanczos vectors,  $Q_m^T M Q_m = I_m$  leads to

$$Q_m^T M K^{-1} M Q_m = T_m \quad (22)$$

That is, the projection of  $M K^{-1} M$  on the subspace with basis  $Q_m$  is a tridiagonal matrix.

This tridiagonal property will be used advantageously in the dynamic response analysis procedure. For this purpose, the equations of motion (Eq. 1) are multiplied by  $M K^{-1}$  with the result:

$$\mathbf{M} \mathbf{K}^{-1} \mathbf{M} \ddot{\mathbf{u}} + \mathbf{M} \mathbf{K}^{-1} \mathbf{C} \dot{\mathbf{u}} + \mathbf{M} \mathbf{u} = \mathbf{M} \mathbf{K}^{-1} \mathbf{f} \quad (23)$$

Before accepting this expression it should be noted that the mass matrix  $\mathbf{M}$  used in dynamic response analyses frequently is singular (because mass coefficients may not be associated with all degrees of freedom), and also that significant information may be lost when a set of equations is multiplied by a singular matrix. Therefore, the use of Eq. 23 may not be justified. However, it is shown in the Appendix that using Eq. 23 instead of Eq. 1 results in the same solution, whether or not  $\mathbf{M}$  is singular.

Now expressing the response in Lanczos coordinates,  $\mathbf{u} = \mathbf{Q}_m \mathbf{x}(t)$  and completing the transformation, Eq. 23 becomes

$$\mathbf{Q}_m^T \mathbf{M} \mathbf{K}^{-1} \mathbf{M} \mathbf{Q}_m \ddot{\mathbf{x}} + \mathbf{Q}_m^T \mathbf{M} \mathbf{K}^{-1} \mathbf{C} \mathbf{Q}_m \dot{\mathbf{x}} + \mathbf{Q}_m^T \mathbf{M} \mathbf{Q}_m \mathbf{x} = \mathbf{Q}_m^T \mathbf{M} \mathbf{K}^{-1} \mathbf{f}$$

Neglecting damping and using Eq. 22 and the orthonormal property of the Lanczos vectors; this may be written in tridiagonal form:

$$\mathbf{T}_m \ddot{\mathbf{x}} + \mathbf{x} = \mathbf{g}_m \quad (24)$$

in which  $\mathbf{g}_m = \mathbf{Q}_m^T \mathbf{M} \mathbf{K}^{-1} \mathbf{f}$ . If Rayleigh damping is used:

$$\mathbf{C} = a_0 \mathbf{M} + a_1 \mathbf{K}$$

the tridiagonal form is still preserved, as follows

$$\mathbf{T}_m \ddot{\mathbf{x}} + [a_0 \mathbf{T}_m + a_1 \mathbf{I}_m] \dot{\mathbf{x}} + \mathbf{x} = \mathbf{g}_m \quad (25)$$

If the more general form of proportional damping given by the Caughey series [8] is adopted:

$$\mathbf{C} = \mathbf{M} \sum_{i=1}^j \mu_i [\mathbf{M}^{-1} \mathbf{K}]^i$$

then the corresponding damping term to be used in Eq. 24 will be of the form

$$\bar{\mathbf{C}}_m = \sum_{i=1}^j \nu_i [\mathbf{T}_m]^{i-1}$$

It should be noted that the damping only approximates the projection of the Caughey damping on the Lanczos subspace but the discrepancy is of no significance in practical cases.

The tridiagonal form of Eq. 24 or 25 is particularly advantageous when the applied load is of

the form of Eq. 17 because then the forcing function can be reduced as follows:

$$\begin{aligned}
 \mathbf{g}_m &= \mathbf{Q}_m \mathbf{M} \mathbf{K}^{-1} \mathbf{f} \\
 &= \mathbf{Q}_m \mathbf{M} \mathbf{K}^{-1} \mathbf{b} \epsilon(t) \\
 &= \mathbf{Q}_m \mathbf{M} \mathbf{q}_1 \beta_1 \epsilon(t) \\
 &= \mathbf{Q}_m \mathbf{M} \mathbf{Q}_m \mathbf{e}_1 \beta_1 \epsilon(t) \\
 &= \beta_1 \mathbf{e}_1 \epsilon(t)
 \end{aligned} \tag{26}$$

where  $\mathbf{e}_1$  is the first column of the identity matrix  $\mathbf{I}_m$ . Eq. 24 then takes the form

$$\mathbf{T}_m \ddot{\mathbf{x}} + \mathbf{x} = \beta_1 \mathbf{e}_1 \gamma(t) \tag{27}$$

which shows that the excitation is applied only in the first of these equations of motion. Of course, the same loading would be applied to the damped system of Eq. 25.

The solution to the equations of motion in the form of Eqs. 24 or 25 may be obtained either by a time-stepping scheme such as the Newmark method [10], or the eigenvectors of  $\mathbf{T}_m$  can be calculated and the response obtained by superposition of the uncoupled equations. The storage demands and the number of operations required for either of these analytical procedures is of the order  $m$ . This is in contrast with the cost of solving an  $n \times n$  eigenproblem in order to employ a standard mode superposition analysis. Moreover, because the Lanczos vectors explicitly recognize the load distribution, fewer Lanczos vectors than mode shapes are needed to obtain a desired level of approximation.

#### Required Number of Vectors

In order to determine how many Lanczos vectors may be required to obtain the desired accuracy in a dynamic analysis it is necessary to assess the errors resulting from a given approximation; or more precisely, to determine the number of vectors required with a tolerable error. The participation factor, which indicates the component of the applied load that contributes to the response of the corresponding Lanczos vector, provides a measure of the significance of that vector in the total response. Defining the  $j$  participation factor as

$$\eta_j = \mathbf{q}_j^T \mathbf{b} \tag{28}$$

the product  $\mathbf{Q}_m^T \mathbf{b} = \mathbf{h}_m$  gives a vector listing the participation factors of all  $m$  Lanczos vectors.

So if the elements of this vector are evaluated sequentially during construction of the Lanczos vectors, their successive values may be judged as a criterion for termination of the Lanczos algorithm.

The values of the participation factors associated with each Lanczos vector may be obtained at negligible cost during the determination of the vector. Multiplying Eq. 16b by  $b^T$  yields

$$b^T r_j = b^T \bar{F}_j - b^T q_j \alpha_j - b^T q_{j-1} \beta_j \quad (29)$$

Noting from Eq. 16a that  $\bar{F}_j = K^{-1}Mq_j$ , and recalling that the starting vector was given by  $\beta_1 q_1 = K^{-1}b$  (Eq. 18), the first term on the right side of Eq. 29 becomes  $b^T K^{-1}Mq_j = \beta_1 q_1^T Mq_j = 0$ . Thus, introducing Eq. 16c on the left side, Eq. 29 can be reduced to

$$\beta_{j+1} b^T q_{j+1} = -b^T q_j \alpha_j - b^T q_{j-1} \beta_j$$

or making use of the definition of the participation factors (Eq. 28), this gives

$$\eta_{j+1} = - \frac{(\alpha_j \eta_j + \beta_j \eta_{j-1})}{\beta_{j+1}} \quad (30)$$

Including this simple scalar calculation with the Lanczos algorithm makes it possible to terminate the calculation when the magnitude of  $\eta_{j+1}$  drops below a specified tolerance value.

### Loss of orthogonality

The simple Lanczos algorithm presented here, involving orthogonalization with only the two preceding vectors at each step, is subject to loss of orthogonality with respect to earlier vectors due to roundoff error. If such errors are not corrected when they reach a critical size, the vectors may become linearly dependent. Paige [11] characterized the way in which orthogonality is lost and provided the theoretical background for avoiding this gradual loss of orthogonality without applying full reorthogonalization at each step. From Paige's work a number of procedures were developed [1,12] which take advantage of the simplicity of the basic Lanczos algorithm while maintaining robust orthogonality.

An algorithm proposed by Simon [12] is adopted here as the simplest means of retaining the necessary level of orthogonality. The vector  $w_j = q_{j+1} M q_j$  is formed to indicate the loss of



orthogonality at any step  $j$ ; that is  $w_j$  contains terms of the order of roundoff, and Simon's algorithm is an inexpensive scheme for updating  $w_j$  at each step of the Lanczos analysis. His formulation for  $w_{j+1}$  may be written

$$\beta_{j+1} w_{j+1} = T_j w_j - \alpha_j w_j - \beta_j w_{j-1} \quad (31)$$

To start this algorithm it is necessary to note that  $w_0 = 0$  and  $w_1 = q_1^T M q_1$ ; then the further values of  $w_j$  are obtained at the cost of only  $5j$  multiplications per step. The magnitude of the elements of this vector demonstrate the trend of the orthogonality condition. When any element of  $w_j$  is excessive,  $q_{j+1}$  is orthogonalized against all the preceding vectors. Then the simple Lanczos algorithm is continued until the vector  $w_j$  at some subsequent step again indicates the need for orthogonalization against all vectors.

### Numerical Results

As an example we use the finite element model of a multistory building frame shown in Fig. 1. The model consist of 170 two dimensional truss elements resulting in a total of 100 degree's of freedom. The material properties were chosen such that the fundamental period is close to 1 second ( $T_1 = 1.082$  sec.). A step-function load is applied at the third story of the structure as shown in the figure. This loading configuration ensures the participation of the higher modes in the response of the structure. It is worth noting that this model also may be viewed as a lattice analogy representation of a plane stress orthotropic elasticity problem, consequently the response to the step function load will contain the effects of a stress wave radiating from the load point. Also it should be noted that the structure was assumed to be undamped in order that the higher mode contributions to the response would not be suppressed.

The Lanczos method was applied to reduce the equations of dynamic equilibrium to the tri-diagonal form. The Newmark step-by-step method then was used to evaluate the response of the reduced problem. Newmark parameters,  $\beta$  and  $\gamma$  were chosen to be  $\beta = \frac{1}{4}$  and  $\gamma = \frac{1}{2}$  to avoid any numerical damping for consistency with the physical assumption and the time step was taken to be  $\Delta t = 0.001$  sec. This requires performing just over 1000 time steps per fundamental period

of the response. The analysis was carried out for various numbers of Lanczos vectors. In figure 2 the displacement response at the top story,  $u$ , is compared with that obtained by applying the Newmark method to the unreduced problem using the same parameters. From figure 2 it is clear that the response of the structure can be obtained accurately with as few as 15 Lanczos vectors. In figure 3 we show the way the displacement  $u$  at time  $t = 3.0$  sec. stabilizes as the number of Lanczos vectors is increased. This convergence behaviour is also typical of the finite element method as the number of elements are increased. This similarity is due to the fact that, like the finite element method, the Lanczos method also minimizes the potential energy. From this plot it can be seen that with only 6 Lanczos vectors the displacement  $u$  is to within 5% and with 20 vectors to within 2% of the solution obtained by the direct method. Figure 4 shows the variation of the Lanczos participation factors,  $\eta_j$ , with the number of Lanczos vectors. It is apparent that as  $\eta_j$  approaches zero the displacement response stabilizes. Therefore as soon as  $\eta_j$  falls below a specified tolerance the Lanczos algorithm can be terminated.

The orthogonality state among the Lanczos vectors was monitored using the algorithm of Eq. 31. In this example, a reorthogonalization step was carried out, on average, every 5 iterations with the first occurring at iteration 6.

### Concluding Remarks

This paper points out that the Lanczos algorithm can be related directly to the inverse iteration method of eigenvector evaluation. The essential concept of the Lanczos procedure is that the successive vectors obtained during inverse iteration become the coordinates used in the dynamic response analysis when they are successively made orthogonal to the previously derived vectors. It is shown here that the orthogonality need be enforced only with respect to the two preceding vectors; the way the Gram-Schmidt orthogonalization is applied automatically ensures orthogonality with all earlier vectors, and this fact permits the equations of motion to be restated in tridiagonal form. Thus the dynamic response may be obtained by step-by-step integration of the simplified equations.

An alternative analysis procedure, of course, would be to derive the vibration mode shapes from the Lanczos coordinates, and then to calculate the response in terms of these uncoupled modal coordinates. However, because the dynamic loading generally is defined as a numerical sequence, the response usually must be obtained by step-by-step method even for a mode superposition solution. Typical time-stepping methods such as those of the Newmark family, requires a total of  $8m + (4w - 2)m$  operations per time step to evaluate the solution to an undamped system of  $m$  equations with its matrices having a half band width  $w$ . When the system is tridiagonal ( $w = 2$ ), as in Eq. 27, the cost per time step becomes  $14m$  operations, and when the system is uncoupled this cost only reduces to  $10m$ . It appears that this additional reduction is not substantial enough to justify the overhead cost of  $3m^3 + 9m^2$  operations required for reducing Eq. 27 to an uncoupled system of equations.

## Appendix

The question that is addressed here is whether the solution to Eq. 23 is equivalent to the solution of the original equation (Eq. 1) when  $M$  is singular. For simplicity the undamped case is considered, and with no loss in generality, the mass and stiffness matrices can be partitioned as follows:

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

where  $K_{12} = K_{21}^T$ . The explicit forms of  $K$  and  $M$  is used to reduce Eq. 1 to

$$M_1 \ddot{u}_1 + K_{11} u_1 + K_{12} u_2 = f_1 \quad (32a)$$

$$K_{21} u_1 + K_{22} u_2 = f_2 \quad (32b)$$

where  $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ . Multiplying Eq. 32b by  $K_{12}K_{22}^{-1}$  and subtracting the result from

Eq. 32a, eliminates  $u_2$  to get

$$M_1 \ddot{u}_1 + (K_{11} - K_{12} K_{22}^{-1} K_{21}) u_1 = f_1 - K_{12} K_{22}^{-1} f_2 \quad (33)$$

The next step is to obtain an expression for  $K^{-1}$  by performing the block  $L^T DL$  factorization of  $K$  (not the usual  $LDL^T$ ) and inverting the result to get

$$K^{-1} = \begin{bmatrix} I & 0 \\ -K_{22}^{-1} K_{21} & I \end{bmatrix} \begin{bmatrix} (K_{11} - K_{12} K_{22}^{-1} K_{21})^{-1} & 0 \\ 0 & K_{22}^{-1} \end{bmatrix} \begin{bmatrix} I - K_{12} K_{22}^{-1} \\ 0 & I \end{bmatrix} \quad (34)$$

When putting this relation and the singular expression for  $M$  in Eq. 13, the first term on the left hand side becomes

$$M K^{-1} M \ddot{u} = \begin{bmatrix} M_1 (K_{11} - K_{12} K_{22}^{-1} K_{21})^{-1} M_1 \ddot{u}_1 \\ 0 \end{bmatrix} \quad (35)$$

and similarly for the right hand side of Eq. 13 yields

$$M K^{-1} f = \begin{bmatrix} M_1 (K_{11} - K_{12} K_{22}^{-1} K_{21})^{-1} (f_1 - K_{12} K_{22}^{-1} f_2) \\ 0 \end{bmatrix} \quad (36)$$

and the first of these equations becomes

$$M_1 (K_{11} - K_{12} K_{22}^{-1} K_{21})^{-1} M_1 \ddot{u}_1 + M_1 u_1 = M_1 (K_{11} - K_{12} K_{22}^{-1} K_{21})^{-1} (f_1 - K_{12} K_{22}^{-1} f_2)$$

Multiplying this equation by  $(K_{11} - K_{12} K_{22}^{-1} K_{21}) M_1^{-1}$  reduces to Eq. 33.

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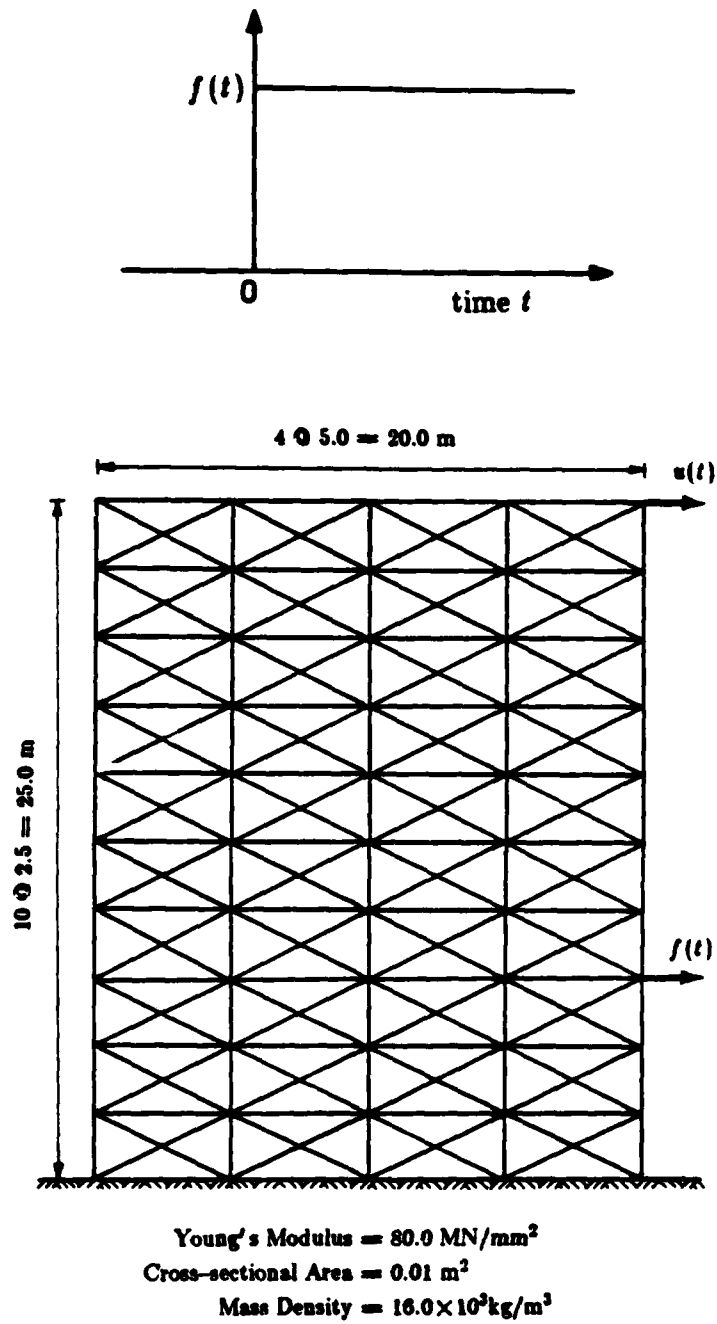


Figure 1. Details of the Truss Example.

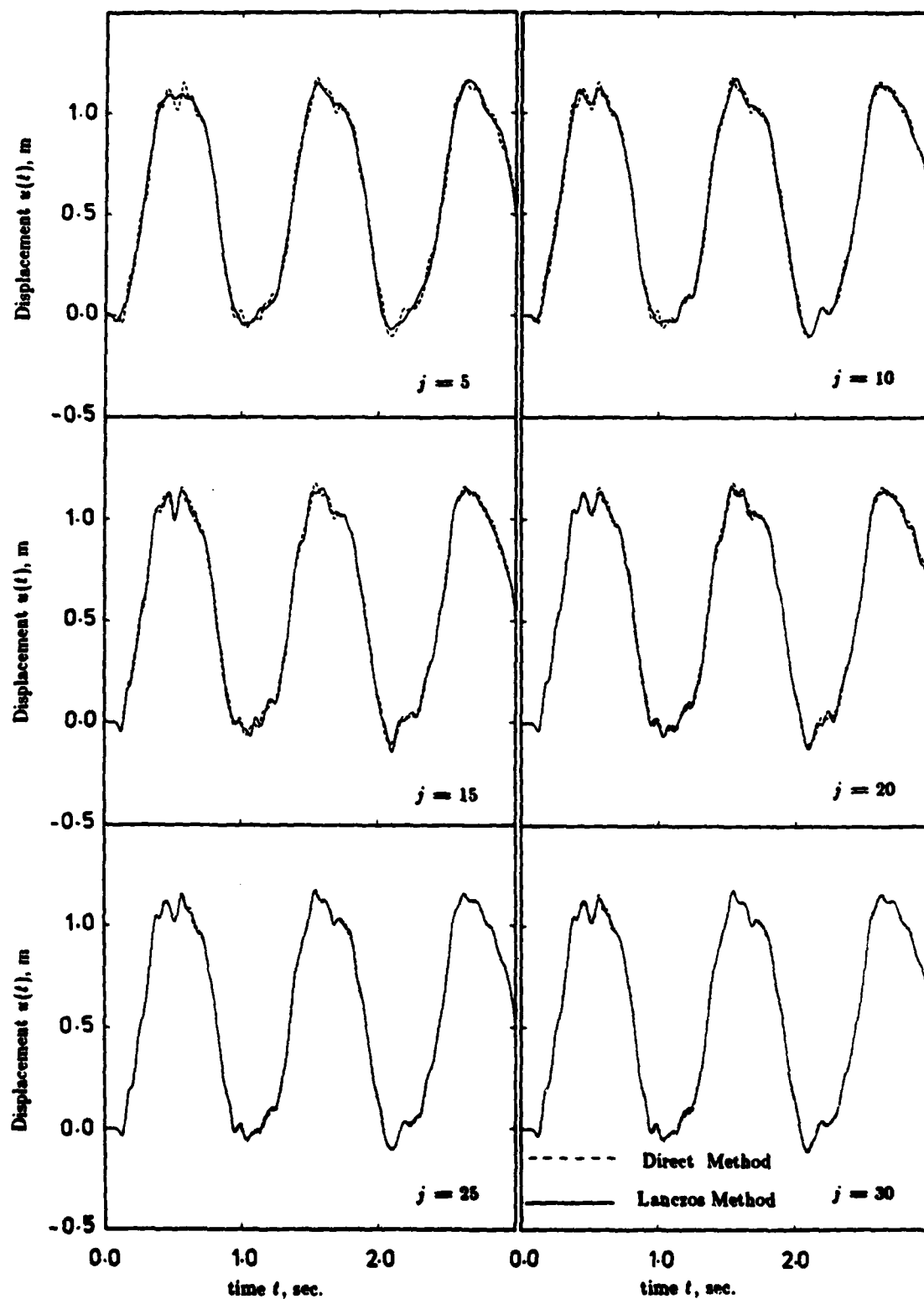


Figure 2. Comparison of the Computed Response Obtained Using Different Number of Lanczos Vectors and Direct Integration.



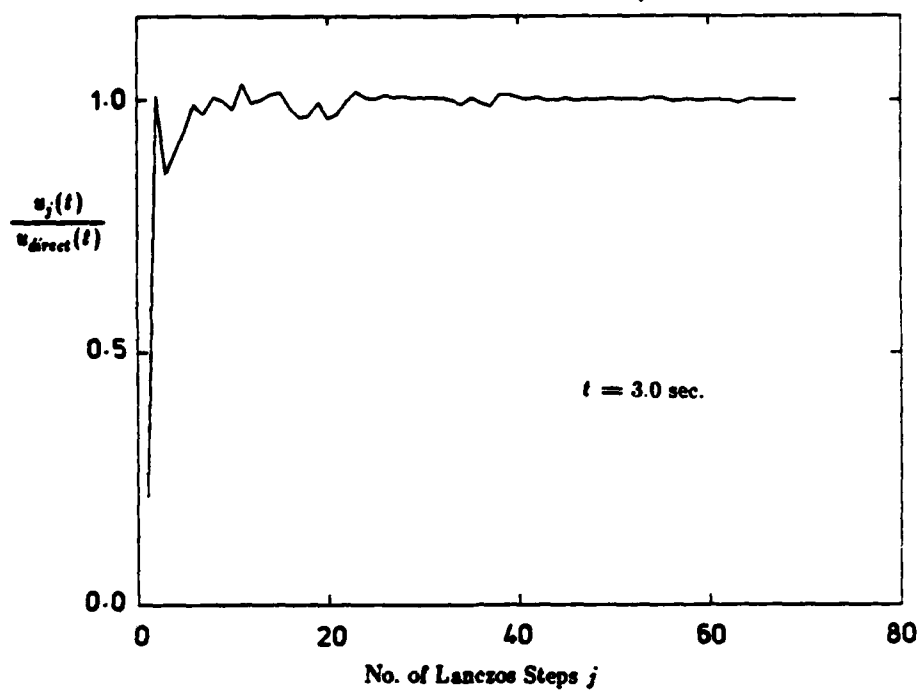


Figure 3. Convergence Pattern of the Response  $u(t)$ .

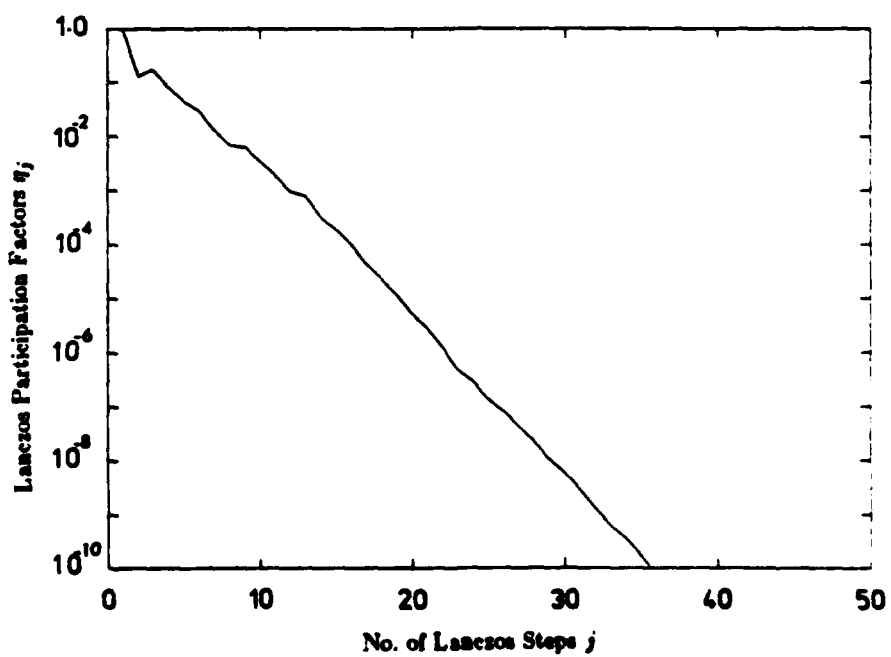
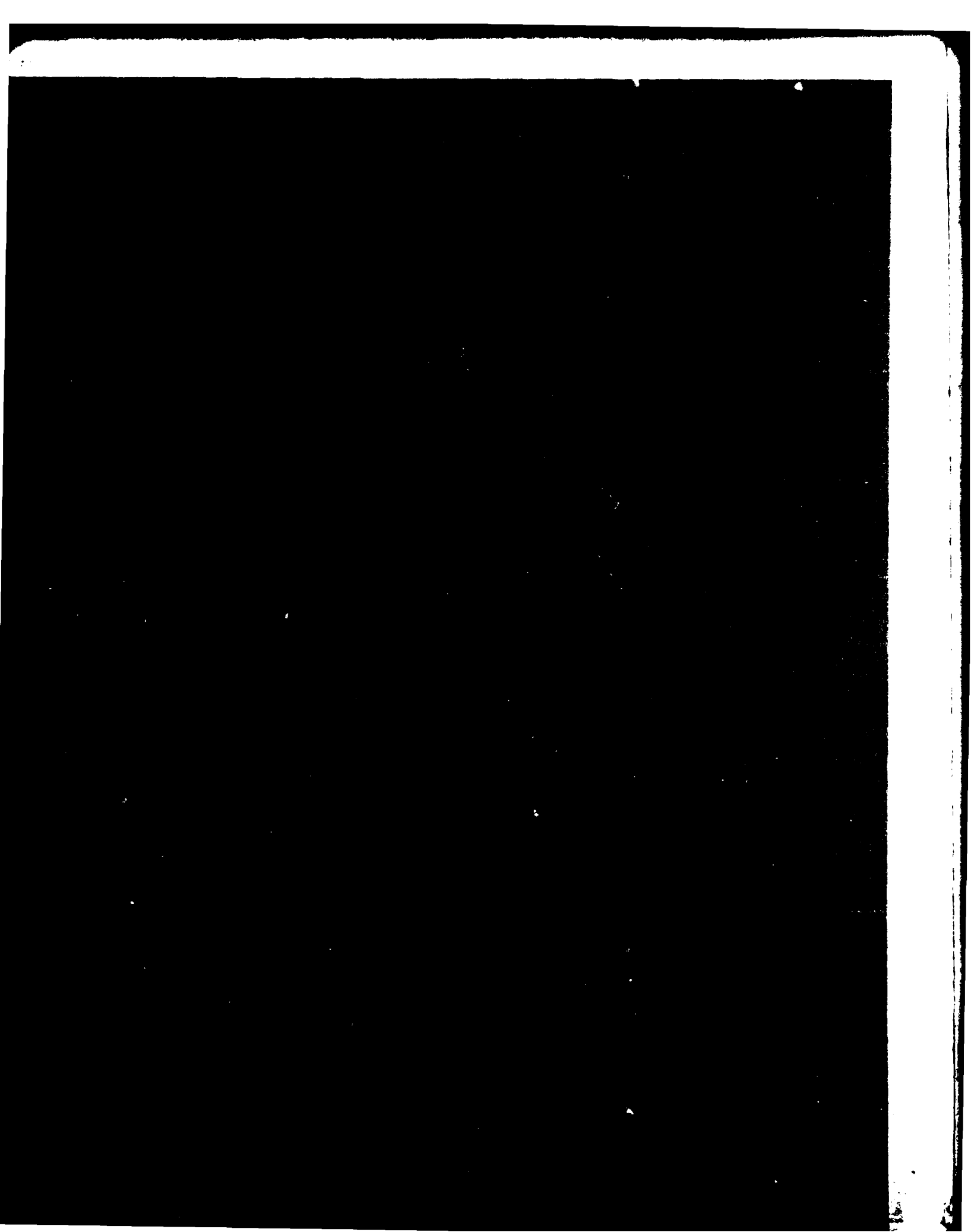


Figure 4. Reduction of the Lanczos Participation Factors with increasing Number of Vectors.



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